ON GENERALIZED FRACTIONAL GABOR TRANSFORM

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ABSTRACT

In this paper we develop the standard definition of Gabor transform considering rotation relation and called it as fractional Gabor transform. Also, we are extending fractional Gabor transform which is a generalization of standard Gabor transform to the distribution of compact support using kernel method. For that we define Testing function space. We first prove that the definition of fractional Gabor transform is authentic. Analyticity theorem, inversion formula is derived.

Keywords: Gabor transform, fractional transform, fractional Gabor transform, testing function space.

Mathematics Subject Classification: 46F12.

1. INTRODUCTION

The fractional Fourier transform has become the focus of many research papers related to several indispensable concepts appearing in diverse areas. Because of its recent application in many fields, including optics and signal processing, the brief account of its application is discussed in Ozactus [7].

The fractional Fourier transform $R^{\propto}$ is an extension of the ordinary Fourier transform which depends on a parameter $\propto$. Extended fractional Fourier transform to the distribution of compact support explained by Bhosale [4], Pao-Yen Lin [8].

The one-dimensional fractional Fourier transform with parameter $\propto$ of $f(t)$ denoted by $R^{\propto}f(t)$ and defined by the integral transform as follows,

$$[R^{\propto}f(t)](\eta) = F_{\alpha}(\eta) = \int_{-\infty}^{\infty} K_{\propto}(t,\eta) f(t)dt$$

Where $K_{\propto}(t,\eta)$ is the kernel given as in Zemanian A.H.

A fractional Gabor transform proposed in Akan A. and Wang Q. [1] is a generalization of the conventional Gabor transform based on the Fourier transform to the Windowed fractional Fourier transform. This transform provides analysis of signals in both the real space and the fractional Fourier transform frequency domain simultaneously [2,3]. This fractional Gabor transform has an additional freedom compared with the conventional Gabor transform. The fractional Gabor transform may offer a usual tool for guiding optimal filter design in the fractional Fourier transform domain in signal processing [13].

Motivated by the above, in this paper we have introduced fractional Gabor transform, which is generalization of standard Gabor transform [5]. The standard Gabor transform given in Pei S.C. and Ding J.J [9].

$$G(u,t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} e^{-i} \quad f(x)dx \tag{1.1}$$

Now we first give the definition of fractional Gabor transform which is rotation of standard Gabor transform given by 1.1

1.1.Definition

The one-dimensional fractional Gabor transform with parameter $\propto$ of $f(x)$ denoted by $G^{\propto}(u,t)$ given by

$$[G^{\propto} f(x)](u) = G^{\propto}(u,t)$$

$$= \sqrt{\frac{1-i\cot\propto}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\frac{(x^2+u^2)}{2}\cot\propto} e^{-\frac{(x-t)^2\csc\propto}{2}} e^{-iux\csc\propto} dx \tag{1.2}$$

Poisson Summation Formula Associated with the Fractional Gabor Transform is proved in [11]. A convolution and product theorem for the Fractional Gabor transform given by [12]. Parseval's Equation and Modulation Property for Fractional Gabor Transform given by [10].

The paper is organized as follows. We first define the fractional Gabor transform which is generalization of standard Gabor transform defined as (1.1) also we have defined generalized fractional Gabor transform in section II. In section III we prove the analyticity theorem. Section IV derived Inversion formula.

Notations and terminology are used as in Zemanian A.H. [14].

2. GENERALIZED FRACTIONAL GABOR TRANSFORM

To define fractional Gabor transform in the generalized sense, we first define the testing function space E as,

2.1. The Testing function space *E*'

An infinitely differentiable complex valued function ψ on R^n belongs to $E(R^n)$ or E if for each compact set $K \subset S_a$ where,

$$S_a = \{x \in R^n, |x| \leq a, a > 0\}, k \in N^n,$$

$$\gamma_{E,k}(\psi) = \sup_{x \in K} |D^k \psi(x)| < \infty.$$

Clearly E is complete and so a Frechet space, also if f is a member of E' (the dual space of E) then we say that f is a fractional Gabor transformable.

2.2. The fractional Gabor transform on E'

For each $u \in R^n$ and $0 \leq \propto \leq \frac{\pi}{2}$, the function $K_\propto(x,u,t)$ belongs to E as a function of x. Hence the fractional Gabor transform of $f \in E'$ can be defined by

$$[G^\propto f(x)](u,t) = G_\propto(u,t) = \langle f(x), K_\propto(x,u,t)\rangle \qquad (2.1)$$

Where,

$$K_\propto(x,u,t) = \sqrt{\frac{1-i\,\text{co}\propto}{2\pi}}\, e^{i\frac{(x^2+u^2)}{2}\cot\propto} e^{-\frac{(x-t)^2\csc\propto}{2}} e^{-iux\csc\propto} \qquad (2.2)$$

Then the right-hand side of (2.1) has a meaning as the application of $f \in E'$ to $K_\propto(x,u,t) \in E$. It can also be extended to $E\ (C^n)$.

$$[G^\propto f(x)](\xi) = G_\propto(\xi) = \langle f(x), K_\propto(x,\xi,t)\rangle \qquad (2.3)$$

For each $\xi \in C^n$, $K_\propto(x,\xi) \in E$ as a function of x.

3. ANALYTICITY THEOREM

We prove following theorem, for discussion of analytic character of generalized fractional Gabor transform.

3.1 Theorem

Let $f \in E'(R^n)$ and let its fractional Gabor transformation be defined by (2.3). Then $[G^\propto f(x)](\xi)$ is analytic on C^n if the $Supp\ f \subset S_a$, where,

$$S_a = \{x : x \in R^n, |x| \leq a, a > 0\}$$

and for $0 < \propto < \pi$, $G_\propto(\xi)$ is differentiable and

$$D_\xi^k G_\alpha(\xi) = \langle f(x), D_\xi^k K_\propto(x,\xi,t)\rangle.$$

Proof:

Let $\xi = (\xi_1, \xi_2, \dots \dots \dots \xi_n) \epsilon C^n$

We prove the result for $k = 1$. The general result follows by induction.

We first prove that

$$\frac{\partial}{\partial \xi_j} G_\propto(\xi) = \langle f(x), \frac{\partial}{\partial \xi_j} K_\propto(x,\xi,t)\rangle.$$

For fixed $\xi_j \neq 0$, choose two concentric circles C and C' with centre at ξ_j and radii r and r_1 respectively, such that $0 < r < r_1 < |\xi_j|$. Let $\Delta\xi_j$ be a complex increment satisfying $0 < |\Delta\xi_j| < r$.

Consider,

$$\frac{G_\propto(\xi_j+\Delta\xi_j)-G_\propto(\xi_j)}{\Delta\xi_j} - \langle f(x), \frac{\partial}{\partial\xi_j} K_\propto(x,\xi,t)\rangle = \langle f(x), \psi_{\Delta\xi_j}(x)\rangle, \tag{3.1}$$

Where

$$\psi_{\Delta\xi_j}(x) = \frac{1}{\Delta\xi_j}\left[K_\propto\left(x,\xi_1,\xi_2,\ldots\ldots\xi_j+\Delta\xi_j\ldots\ldots\xi_n,t\right) - K_\propto(x,\xi,t)\right] - \frac{\partial}{\partial\xi_j}K_\propto(x,\xi,t).$$

For any fixed $x \in R^n$ and any fixed integer$k = (k_1, k_2, \ldots, k_n) \in N^n$,

$D_x^k K_\propto(x,\xi,t) = D_x^k[C_{1\propto} exp\{C_{2\propto}[i[(x^2+\xi^2)\cos\alpha - 2\xi x] - (x-t)^2]\}]$,

Where

$C_{1\propto} = \sqrt{\frac{1-i\cot\propto}{2\pi}}$, $C_{2\propto} = \frac{1}{2\sin\propto}$.

$\therefore D_x^k K_\propto(x,\xi,t) = K_\propto(x,\xi,t)\sum_{j=0}^{n} C_{\propto,j}[i(2x\cos\propto - 2\xi) - 2(x-t)]^{k-2j}$,

Where k is order of derivative,

n depends on order of derivative,

and $C_{\propto,j}$ are constants depend on $\propto, k, j$.

Since for any fixed $x \in R^n$ and fix integers k and $\propto$ ranging from 0 to π, $D_x^k K_\propto(x,\xi,t)$ is analytic inside and on C', we have by Cauchy integral formula,

$$D_x^k\psi_{\Delta\xi_j}(x) = \frac{1}{2\pi i} D_x^k \int_{C'} K_\propto\left(x,\bar{\xi},t\right)\left[\frac{1}{\Delta\xi_j}\left(\frac{1}{z-\xi_j-\Delta\xi_j} - \frac{1}{z-\xi_j}\right) - \frac{1}{(z-\xi_j)^2}\right]dz,$$

$$= \frac{\Delta\xi_j}{2\pi i}\int_{C'} \frac{M(x,\bar{\xi},t)}{(z-\xi_j-\Delta\xi_j)(z-\xi_j)^2}dz,$$

Where $\bar{\xi} = (\xi_1, \ldots\ldots \xi_{j-1}\, z\, \xi_{j+1} \ldots\ldots \xi_n)$

and $D_x^k K_\propto\left(x,\bar{\xi},t\right) = M\left(x,\bar{\xi},t\right)$.

But for all $z \in C'$ and x restricted to compact subset of R^n, $0 < \propto < \pi$, $M(x,\bar{\xi},t)$ is bounded by a constant K and $|z-\xi_j-\Delta\xi_j| > r_1 - r > 0$, $|z-\xi_j| = r_1$.

Therefore we have,

$$\left|D_x^k\psi_{\Delta\xi_j}(x)\right| \le \frac{|\Delta\xi_j|K}{(r_1-r)r_1}.$$

Thus as $|\Delta\xi_j| \to 0$, $\left|D_x^k\psi_{\Delta\xi_j}(x)\right| \to 0$ uniformly on the compact subset of R^n.

Therefore it follows that $\psi_{\Delta\xi_j}(x)$ converges in $E(R^n)$ to zero. Since f∈ E' we conclude that (3.1) also tends to zero. Therefore $M_\propto(\xi)$ is differentiable with respect to ξ_j. But this is true for all $j = 1,2, \dots .n$.

Hence $M_\propto(\xi)$ is analytic on C^n and $D_\xi^k G_\alpha(\xi) = \langle f(x), D_\xi^k K_\propto (x, \xi, t)\rangle$.

4. INVERSION FORMULA

For generalised fractional Gabor transform we established the fundamental result that is "Inversion formula" in this section. It is possible to recover the function f by means of the inversion formula.

$f(x) = \frac{1}{\pi i}\int_{-\infty}^{\infty} G_\propto (u)\overline{K_\propto (x, u, t)}\, du$, where

$\overline{K_\propto (x, u, t)} = \left(\frac{\csc\propto}{2}\right)\left(\frac{1-i\,\text{co}\ \propto}{2\pi}\right)^{-1/2} e^{-i\frac{(x^2+u^2)}{2}\cot\propto} e^{\frac{(x-t)^2 \csc\propto}{2}} e^{iux\csc\propto}$.

Proof:

The one-dimensional fractional Gabor transform is given by

$$[G^\propto f(x)](u, t) = G_\propto(u, t) = \int_{-\infty}^{\infty} f(x) K_\propto (x, u, t) dx \quad (4.1)$$

Where the kernel is,

$$K_\propto (x, u, t) = \sqrt{\frac{1 - i\cot\propto}{2\pi}}\, e^{i\frac{(x^2+u^2)}{2}\cot\propto} e^{\frac{-(x-t)^2 \csc\propto}{2}} e^{-iux\csc\propto}$$

$$= C_{1\propto}\, e^{i(x^2+u^2)C_{2\propto}\cos\alpha} e^{-(x-t)^2 C_{2\propto}}\, e^{-iux\quad 2\propto}.$$

where

$C_{1\propto} = \sqrt{\frac{1-i\,\text{co}\ \propto}{2\pi}}$, $C_{2\propto} = \frac{1}{2\,\text{si}\ \propto}$.

From (1.2)

$$e^{-ic_{2\propto}u^2\cos\propto} G_\propto(u, t) = \int_{-\infty}^{\infty} g(x) e^{-(x-t)^2 C_{2\propto}} e^{-iux2C_{2\propto}}\, dx$$

Where

$$g(x) = C_{1\propto}\, f(x) e^{iC_{2\propto}x^2\cos\propto} \quad (4.2)$$

$$e^{-ic_{2\propto}u^2\cos\propto} G_\propto(u, t) = G[g(x)](2C_{2\propto}u, t) \quad (4.3)$$

Where,

$G[g(x)]$ is Gabor transform of $g(x)$ with argument $\eta = 2C_{2\propto}u$ (say) (4.4)

Thus $e^{-ic_{2\propto}u^2\cos\propto}G_{\propto}\left(\frac{\eta}{2C_{2\propto}},t\right)=G[g(x)](\eta,t)$,

Invoking Gabor inversion, we can write,

$g(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}G(\eta,t)e^{(x-t)^2C_{2\propto}}e^{iux2C_{2\propto}}\,d\,\eta,$

Where $G(\eta,t)=e^{-ic_{2\propto}\frac{\eta^2}{4C_{2\propto}{}^2}\cos\propto}G_{\propto}\left(\frac{\eta}{2C_{2\propto}},t\right)$

Putting the value of $g(x)$ from (4.2) and simplifying we get inversion formula,

$$f(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty}G_{\alpha}(u)\overline{K_{\alpha}(x,u,t)}\,du$$

Where,

$\overline{K_{\alpha}(x,u,t)}=\left(\frac{\csc\propto}{2}\right)\left(\frac{1-i\cot\propto}{2\pi}\right)^{-1/2}e^{-i\frac{(x^2+u^2)}{2}\cot\propto}e^{\frac{(x-t)^2\csc\propto}{2}}e^{iux\csc\propto}.$

5. DISCUSSION AND CONCLUSION

In this paper we have discussed fractional Gabor transform in the generalized sense. Analyticity theorem proved for fractional Gabor transform. Also, inversion formula is proved for this transform.

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