CO-REGULAR SPLIT DOMINATION IN GRAPHS

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ABSTRACT: *In this paper, we introduce the new concept in domination theory. A dominating set* $D \subseteq V(G)$ *is a coregular split dominating set if the induced subgraph* $< V - D >$ *is regular and disconnected. The minimum cardinality of such a set is called a coregular split domination number and is denoted by* $\gamma_{crs}(G)$. *Also we study the graph theoretic property of* $\gamma_{crs}(G)$ *and many bounds were obtained interms of* G *and its relationship with other domination parameters were found.*

KEYWORDS: Dominating set /Split domination / Total domination/ Regular domination / Coregular split domination.

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1. INTRODUCTION

All graphs considered here are simple and without isolated vertices. Let $G = (V, E)$ be a graph with $|V| = P$ and $|E| = q$. We denote $< V - D >$ to denote the subgraph induced by the set of vertices of D and $N(v)$ and $N[v]$ denote the open and closed neighborhood of a vertex v, respectively. Let $\deg(v)$ be the degree of a vertex v and as usual $\delta(G)$ the minimum degree and $\Delta(G)$ maximum degree. In general we follow the notation and terminology of Harary [2].

A vertex cover in a graph G is a set of vertices that covers all the edges of G. The vertex covering number $\propto_o(G)$ is a minimum cardinality of a vertex cover in G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G .The edge covering number $\alpha_1(G)$ is a minimum cardinality of a edge cover in G.

A line graph $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

A block graph $B(G)$ is the graph whose set of vertices is the union of set of blocks of G in which two vertices are adjacent if and only if the corresponding blocks of G are adjacent.

A graph is r-regular when all its vertices have degree r, namely $\triangle(G) = \delta(G) = r$. We begine with standard definitions from domination theory.

A set $D \subseteq V$ is a dominating set of G if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a dominating set in G is the domination number and denoted by $\gamma(G)$. For comprehensive work on the subject has been done in [3].

A dominating set $D \subseteq V(G)$of a graph $G = (V, E)$ is called a connected dominating set if the induced subgraph $< D >$ is connected. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set of G see [4].

A dominating set $D \subseteq V(G)$ is a total dominating set of a graph G if the induced graph $< D >$ does not contain an isolated vertex. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G. The total domination in graph was introduced by Cockayne et al.[1] in 1980.

A dominating set $D \subseteq V(G)$ is a cotatal dominating set if the induced subgraph $< V - D >$ has no isolated vertices. The cototal domination number $\gamma_{ct}(G)$ of G is the minimum cardinality of cototal dominating set of G.

A dominating set $D$of G is called split dominating set if the induced subgraph $< V - D >$is disconnected. The split domination number is $\gamma_s(G)$ of a graph G is the minimum cardinality of a split dominating set of G.

A dominating set D of G is called strong split dominating set of G if $< V - D >$ is totally disconnected with at least two vertices. The strong split domination number $\gamma_{ss}(G)$ of a graph G is the minimum cardinality of a strong split dominating set of G[5].

A dominating set D of G is a global dominating set if it is also dominating set of $\bar{G}$. A minimal cardinality of global dominating set is the global domination number and is denoted by $\gamma_g(G)$**[7].**

A dominating set D of $L(G)$ is a global dominating set if it is also dominating set of $L(\bar{G})$. A minimal cardinality of D is a global domination number of $L(G)$ and denoted by $\gamma_{gl}(G)$ see[6].

2. RESULTS

We develope the following results for some standard graphs.

Theorem 1: a] For any path p_p with $p \geq 2$ vertices,

$$\gamma_{crs}(p_p) = \left\lfloor \frac{p}{2} \right\rfloor.$$

b] For any star $k_{1,p}$ with $p \geq 2$ vertices,

$$\gamma_{crs}(k_{1,p}) = 1.$$

Theorem 2: For any connected (p,q) graph G with $p \geq 3$, then

$$\gamma_{crs}(G) + \gamma(G) \leq p .$$

Proof: Let $V_1 = \{ v_1, v_2, \dots \dots \dots \dots, v_n\} \subseteq V(G)$ be the set of all non end vertices in G. The $V'_1 \subseteq V_1$ forms a $\gamma - set$ of G. Let $V_2 = \{ v_1, v_2, \dots \dots \dots \dots, v_m\} \subseteq V_1$ where every $v_i \in V_2$ is adjacent to end vertices. Further $V_3 = \{ v_1, v_2, \dots \dots \dots, v_k\} \subseteq V_1$ be the set of vertices with maximum degree. Suppose $< V(G) - V_2 \cup V_3 >$ is disconnected and $\forall\, v_i \epsilon\, [V(G) - \{V_2 \cup V_3\}]$ has same degree $< V_2 \cup V_3 >$ forms a $\gamma_{crs} - set$. Otherwise there exists a set $A = \{ v_1, v_2, \dots \dots \dots \dots, v_k\}$ of vertices which are neighbors of some vertices in V_3 . Now $< V(G) - V_2 \cup V_3 \cup A >$ is disconnected with isolated vertices of cardinality at least two. Then $|V_2 \cup V_3 \cup A| + |V_1| \leq V(G)$, which gives $\gamma_{crs}(G) + \gamma(G) \leq p$.

The following result gives an upper bounds for $\gamma_{crs}(G)$ in terms of γ_c and γ_t of G.

Theorem 3: For any connected (p,q) graph G with ≥ 3 , then

$$\gamma_{crs}(G) \leq \gamma_c + \gamma_t \text{ and } G \neq W_p \ \ (P > 5).$$

Proof: Let $V = \{ v_1, v_2, \dots \dots \dots \dots, v_k\}$ be the vertex set of G. Now for the graph $G \neq W_p$ with $p \geq 4$, suppose $p \leq 4$ the $\gamma_c + \gamma_t = 3 = \gamma_{crs}(G)$ and result holds. Further if $P > 5$, $|\gamma_c + \gamma_t| = 3$ and $\gamma_{crs}[W_p] = \frac{p}{2} + 1 > |\gamma_c + \gamma_t|$. Hence $G \neq W_p$ with $P > 5$. Now let $A = \{ v_1, v_2, \dots \dots \dots \dots, v_n\} \subseteq V(G)$ suppose for every $v \in \{V(G) - A\}$ is adjacent to at least one vertex of A. If $< A >$ has no isolated vertices then A itself is a total dominating set of G. Otherwise let $v \in \{V(G) - A\}$ and if $\{A\} \cup \{v\}$ has no isolated vertex. Clearly $\{A\} \cup \{v\}$ is a minimal total dominating set of G. Let $A_1 = \{ v_1, v_2, \dots \dots \dots \dots, v_n\}$ be the set of all end vertices in G. $A_2 = \{ V(G) - A_1\}$ be the set of all nonend vertices in G. Suppose there exists a minimal set of vertices such that $N[v_i] = V(G)\ \forall\, v_i \in A_2$, $1 \leq i \leq n$ then A_2 forms a minimal dominating set of G. Further if $A_2 = \{ V(G) - A_1\}$ has exactly one component then A_2 itself is a connected dominating set of G. Suppose A_2 has more than one component then attach the minimum set of vertices. $S' = A_2 \cup \{u, w\}$ which are in $u - w$ path, $\forall\, u, w \in \{V(G) - A_2\}$. Hence $S'$is a minimal connected dominating set of G. Further let $A_2 = \{v_1, v_2, \dots \dots \dots, v_i\}$ be the set of all nonend vertices suppose there exists a minimal dominating set S such that the distance between the two vertices of S is at least two clearly there exists more than one component and each component in $< V - S >$ is regular forms $\gamma_{crs} - set$. Thus $|S| \leq |A_2| + |A|$. Hence $\gamma_{crs}(G) \leq \gamma_c + \gamma_t$.

Now the next theorem gives lower bound on the coregular split domination number of graph (G).

Theorem 4: For any connected (p,q) graph G with $p \geq 3$, then

$$\gamma_{crs}(G) \geq \gamma_{gl}(G) - 1.$$

Proof: Let $E = \{ e_1, e_2, \dots \dots \dots \dots, e_n\}$ be the set of edges in G. Now consider $E_1 = \{ e_1, e_2, \dots \dots \dots, e_k\} \subseteq E(G)$ be the set of edges with maximum edge degree and $E_2 = \{ e_1, e_2, \dots \dots \dots, e_j\} \subseteq E(G)$ be the set of edges with minimum edge degree. Suppose $E'_1 \subseteq E_1$ and $E'_2 \subseteq E_2\ \ \forall\, v \in [V[L(G)] - \{E'_1 \cup E'_2\}]$ is adjacent to at least one vertex of $\{E'_1 \cup E'_2\}$ and $\{\overline{E'_1 \cup E'_2}\}$. Since each edge of G is a vertex in $L(G)$, then $\{E'_1 \cup E'_2\}$ is a global dominating set of $L(G)$. Further let $D = \{v_1, v_2, \dots \dots \dots \dots, v_n\}$ be the set of vertices in G, such that $[V(G) - N(D)]$ is regular and which gives more than one component. Then D

forms a minimal coregular split dominating set of G. Thus $|D| \geq |E'_1 \cup |E'_2| - 1|$ hence $\gamma_{crs}(G) \geq \gamma_{gl}(G) - 1$.

Theorem 5: For any connected (p, q) graph G with $P \geq 3$,then

$$\gamma_{crs}(G) \geq q - \propto_1 (G) + \gamma_g(G) - 1.$$

Proof: Let $A = \{v_1, v_2, v_3, \dots \dots, v_l\}$ be set of all nonend vertices in G. Let $B_1 = \{v_1, v_2, \dots \dots, v_m\} \subseteq A$ be a set of vertices with maximum degree. $B_2 = \{v_1, v_2, \dots \dots v_n\} \subseteq A$ be set of vertices with minimum degree in G.The distance between two vertices of B_1 and B_2 is at most 2. Hence $\{B_1 \cup B_2\}$ is γ- set if $[V(G) - \{B_1\} \cup \{B_2\}]$ disconnected and having vertices with same degree forms a $\gamma - set$. Let $B = \{e_1, e_2, \dots \dots, e_n\}$ be the set of all end edges. Suppose $B' = \{e_1, e_2, \dots \dots, e_k\} \subseteq E(G) - B$ be the set of edges such that dist $(e_i, e_j) \geq 2$ $1 \leq i \leq n$, $1 \leq j \leq k$, then $B \cup F$,where $F \subseteq B'$ be the minimal set of edges which covers all the vertices in G, such that $|B \cup F| = \propto_1 (G)$. Further let $S = \{v_1, v_2, \dots \dots, v_p\} \subseteq V(G)$ and $S \subseteq V(\bar{G})$. If $N[S] = V(\bar{G})$. Then S is dominating set for G and $(\bar{G})$. Therefore S forms a global dominating set of G. Now, we have $|B_1 \cup B_2| \leq q - |B \cup F| + |S| - 1$, which gives $\gamma_{crs}(G) \geq q - \propto_1 (G) + \gamma_g(G) - 1$.

We establish the relationship between, split domination total domination with coregular split domination number in the following theorem.

Theorem 6: For any connected (p, q) graph G with γ_{crs} is 1 −regular then

$$\gamma_{crs}(G) \leq \gamma_s(G) + \gamma_t(G) - 1 \text{ and } G \neq W_p \ (P > 5).$$

Proof: Let $A_1 = \{v_1, v_2, v_3, \dots \dots, v_n\} \subseteq V(G)$ be the set of all end vertices in G and $A'_1 = V(G) - A_1$. Suppose there exists vertex set $F \subset A'_1$ such that $D = [V(G) - F]$ is a dominating set of G. Hence $< D >$ has more than one component with same degree than D forms a $\gamma_{crs} - set$. Suppose there exists set of vertices $C \subseteq A_1{}'$ where $C \cup A_1$ covers all vertices in G and if the subgraph $< V(G) - \{C \cup A_1\} >$ does not containany isolated vertex $C \subset A_1$ itself is a cototal dominating set of G.Otherwise if there exists a vertex $v \in [V(G) - \{C \cup A_1\}$ with $\deg(v) = 0$. Then $C \cup A_1 \cup \{v\}$ forms a minimal $\gamma_{ct} - set$ of G. Further let $B' = \{v_1, v_2, \dots \dots, v_k\} \subseteq V(G)$ be the set all nonend vertices in G. Then $B' \subseteq A_1{}'$ forms a minimal $\gamma - set$ of G. If $< V - D >$ is disconnected then B' forms a split dominating set of G.Hence $|D| \leq |B'| + |C| \cup A_1 \cup \{v\} - 1$ and $\gamma_{crs}(G) \leq \gamma_s(G) + \gamma_t(G) - 1$.

Theorem 7: For any non-trivial tree T with $p \geq 2$, then $\gamma_{crs}(T) = \alpha_0(T)$ if and only if γ_{crs} is zero regular.

Proof : Suppose $\gamma_{crs}(T) = \alpha_0(T)$ and $\gamma_{csr} - set$ is not zero regular. Let $D = \{v_1, v_2, \dots \dots \dots, v_n\}$be a dominating set of T such that the distance between two vertices of D be at most three. If $< V - D >$ is disconnected we consider the following cases.

Case1: Assume there exists at least one edge $e \in V(T) - D$ which is a component of disconnected $< V(T) - D >$. Then γ_{crs} is not zero regular, a contradiction.

Case2: Assume there exists a vertex $v \in \gamma_{crs} - set$ and $v \notin \alpha_0 - set$. Then there exists $N(v) = u$. Such that an edge $uv \in \{V(T) - D\}$ a contradiction.

Conversly, suppose $\gamma_{crs}(T) = \alpha_0(T)$, and $\gamma_{csr}(T)$ is zero regular . Let $D = \{v_1, v_2, \dots \dots \dots \dots, v_n\}$ be a set of vertices such that the distance between two vertices of D be at most two. Hence $N(u) \cup N(v) = \varphi, \forall\, u, v \in D$ and edge of T covered by the set D. Clearly $|D| = \alpha_0(T)$ since D is minimal dominating set of T and $< V - D >$ is disconnected with deg(v)=0 $\forall\, v \in < V - D >$. Then D is also $\gamma_{crs} - set$ which is zero regular. Hence $\gamma_{crs}(T) = \alpha_0(T)$.

In the following Theorem , we establish the upper bound for $\gamma_{crs}(T)$ interms of vertices of graph G.

Theorem 8: For any non-trivial tree T with $p \geq 2$, then $\gamma_{crs}(T) \leq p - m$. Where m is the number of end vertices in T.

Proof : Let $A = \{v_1, v_2, \dots \dots \dots \dots, v_m\} \subseteq V(T)$ be the set of all end vertices in T with $|A| = m$. Let $D = \{v_1, v_2, \dots \dots \dots \dots, v_n\}$ be a dominating set of T.Such that the distance between two vertices of D is at most three. If $< V - D >$ has more than one component. Then vertices of each component have same degree and all component are also have same degree. Then D is $\gamma_{crs} - set$ of a tree T. So that $|D| = p - |A|$ and gives $\gamma_{crs}(T) \leq p - m$.

Theorem 9: For any non-trivial tree T with $p \geq 2$, then $\gamma_{crs}(T) = \gamma_{ss}(T)$.

Proof: Let $H_1 = \{v_1, v_2, v_3, \dots \dots, v_l\}$ be set of all vertices in $V(T)$. Let $H_2 = \{v_1, v_2, v_3, \dots \dots, v_m\}$ be set of all nonend vertices adjacent to end vertices. $H_3 = \{v_1, v_2, v_3, \dots \dots, v_n\}$ be set of all nonend vertices which are not adjacent to end vertices. Let there exists $H'_3 \subseteq H_3$ such that $D = \{H_2\} \cup \{H'_3\} \subseteq V(T)$. Where $\forall\, v_i \in V(T) - D$ is adjacent to at least one vertex of D. Hence D is a minimal dominating set of G. Further if $\forall\, v_i \in < V - D >$ $\deg(v_i) = 0$ with at least two vertices. Hence D is a $\gamma_{crs} - set$ of G. Simillarly by definition of strong split dominating set the subgraph $< V - D >$is a null set with at least two vertices . Hence D is also a $\gamma_{ss} - set$ of G. Clearly $\gamma_{crs}(T) = \gamma_{ss}(T)$.

Further if there exists a set $E = \{e_1, e_2, \dots \dots \dots \dots, e_j\}$ be edges in $< V - D >$ and each component of $V - D$ is K_2. Then D is a $\gamma_{crs} - set$ but not $\gamma_{ss} - set$. For equality if $A = \{v_1, v_2, \dots \dots \dots \dots, v_k\}$ be the set of vertices which are $N(v_m)$, $\forall\, v_m \in B$ where $B = \{v_1, v_2, v_3, \dots \dots, v_l\}$ such that $\{A\} \cup \{B\}$ forms the component as K_2 in $< V - D >$. Then $\forall v_i \in [\{V - D\} - \{A\}]$ or $[\{V - D\} - \{B\}]$ is an isolate. Thus either $\{D\} - \{A\}$ or $\{D\} - \{B\}$ is a $\gamma_{crs} - set$ and also a $\gamma_{ss}(T) - set$ of a tree. Hence $\gamma_{crs}(T) = \gamma_{ss}(T)$.

Theorem 10: For any non-trivial tree T with $p \geq 3$, then

$$\gamma_{crs}(T) + 3 \geq \left\lfloor \frac{q - \gamma_c}{2} \right\rfloor.$$

Proof: Let $V = \{v_1, v_2, \dots \dots \dots \dots, v_l\}$ be vertex set of T and $E = \{e_1, e_2, \dots \dots \dots \dots, e_m\}$ be edge set of T. And $A_1 = \{v_1, v_2, v_3, \dots \dots, v_m\} \subseteq V(T)$ be set of all nonend vertices which are not adjacent to end vertices. If the distance between the two vertices of A_1 and A_2 is at most 2. Suppose there exists a set $A_2{}' \subseteq A_2$ hence $S = [V(T) - \{A_1 \cup A_2{}'\}]$ is a dominating set of T with the property that $< S >$ is totally disconnected . Then S is a $\gamma_{crs} - set$ of T. Let $H = \{A_1 \cup A_2\}$ and $\forall\, v_i \in V(T) - H$ is adjacent to at least one vertex of H then H is dominating set of T and $< H >$ is connected. Hence H is $\gamma_c - set$ of a tree T. Since every vertex of $\gamma_c - set$ is incident with the edges of T then $(E - H)/2 \leq \{S + 3\}$, implies that $|S| + 3 \geq \left\lfloor \frac{|E| + |H|}{2} \right\rfloor$ and gives , $\gamma_{crs}(T) + 3 \geq \left\lfloor \frac{q - \gamma_c}{2} \right\rfloor$.

Next theorem gives upper bound for $\gamma_{crs}(T)$.

Theorem11: For any non-trivial tree T with $p \geq 3$, then

$$\gamma_{crs}(T) \leq \gamma_t[B(T)] + \delta(G).$$

Proof: Let $V_1 = \{v_1, v_2, v_3, \dots \dots, v_n\}$ be the set of end vertices of $V(T)$. $V_2 = \{v_1, v_2, v_3, \dots \dots, v_m\}$ be the set of vertices adjacent to V_1 there exists $V_3 = \{V(T) - V_1 \cup V_2\}$ then $S = \{V_2 \cup V_3\}$ is a minimal dominating set of T. Suppose there exists a $N(V_3) \cap N(V_2) = \emptyset \; \forall \; V_2, V_3 \in S$. Hence each edge of Tcovers by the set S and $< V - S >$ is disconnected such that $\deg(v_i) = 0 \; \forall \; v_i \in < V - S >$ then S is a $\gamma_{crs} - set$ which is zero regular. Further let D^n be dominating set of block graph $B(T)$ of a tree T and $A_1 = V[B(T) - D^n]$ such that $D_1 \subseteq A_1$ and $< D_1 \cup D^n >$ has no isolated vertex . Then $\{D_1 \cup D^n\}$ is $\gamma_t - set$ of T. Let v be a point of minimum degree $\delta(T)$. Hence $|S| \leq |D_1 \cup D^n| + |v|$ which gives, $\gamma_{crs}(T) \leq \gamma_t[B(T)] + \delta(G)$.

In the following two lemmas we have the sharp bound attained to γ_{crs} by considering each block of G which is complete graph K_m and K_n.

Lemma 1. If G has exactly one nonend block K_n and all vertices of K_n are incident with blocks which are K_m with $m \geq n$ (or) $m < n$. Then $\gamma_{crs} = n$.

Proof: Let K_n be a nonend block of G with vertex set $D = \{v_1, v_2, \dots \dots \dots \dots, v_n\}$. Suppose all vertices of K_n are incident with blocks which are K_m. We consider the following cases.

Case1: Suppose each vertex of K_n is incident with L number of blocks which are complete graphs K_m with $m \geq n$. Then D is a dominating set of G. Also the induced subgraph $< V(G) - D >$ is disconnected and $m - 1$ regular. Hence $|D| = \gamma_{crs}(G)$, which is also equal to n. Clearly $\gamma_{crs} = n$.

Case2: Suppose each vertex of K_n is a cut vertex and incident with L number of blocks which are K_m with $m < n$. Then the induced subgraph $< V(G) - D >$ is again disconnected and $m - 1$ regular. Since $\forall \; v_i \in D$ is adjacent to at least one vertex of $V(G) - D$, then D is a $\gamma_{crs} - set$ of G and $|D| = n$. Clearly $\gamma_{crs} = n$.

From the above lemma we concluded that, if there exists at least one block which is either K_{m-1} or K_{m+1} in L number of blocks . Then there does not exists $\gamma_{crs} - set$.

Lemma 2: If G has exactly one cut vertex C incident with blocks which are K_n , $n \geq 2$, then $\gamma_{crs} = C$.

Proof: Suppose G has exactly one cut vertex v which is incident with m number of $K_n (n \geq 2)$ blocks. Then every vertex of $\{G - V\}$ is adjacent to v.Thus $\{v\}$ is a $\gamma - set$ of G and $< G - V >$ is disconnected with m numbere of K_{n-1} blocks. Hence each component of $< G - V >$ is K_{n-1} regular and $\{v\}$ is a $\gamma_{crs} - set$ of G. Since v is a cu vertex then $\gamma_{crs} = C$.

Theorem12: For any graph G with C cut vertices $\gamma_{crs} = C$ if and only if G has exactly one nonend block K_n incident with complete blocks which are K_{n-c+1}.

Proof: Suppose $\gamma_{crs} = C$. Let $H = \{B_1, B_{2,} \dots \dots \dots, B_n\}$ be the set of n blocks of G. Let $A_1 = \{B_1, B_{2,} \dots, B_p\}$ be the end blocks in G. Such that $K_n = H - A_1$ which is nonend block of G. Let $\{v_1, v_2, \dots \dots \dots \dots, v_n\} = V[K_n]$. Suppose $L_1 = \{v_1, v_2, v_3, \dots \dots, v_i\} \subseteq V[K_n]$ be the set of cut vertices. We consider the following cases. Let D be a $\gamma_{crs} - set$ of G.

Case1: Suppose $|L_1|$ cut vertices are incident with blocks which are K_{n-c}. Then L_1 is dominating set of G. But $< V(G) - L_1 >$ is not regular. Hence $\gamma_{crs} = L_1$, contradiction.

Case 2: Suppose $\{v_1, v_2, \dots \dots \dots \dots, v_n\} \in L_1$ are incident with K_{n-c+2} blocks. Then $\{L_1\}$ is a dominating set of G. Further $< V(G) - \{v_1, v_2, \dots \dots \dots \dots, v_n\} >$ is not a regular, a contradiction.

Case 3: Suppose the number of cut vertices $|L_1| > |V[K_n] - L_1|$. Then L_1 is a dominating set of G and $< V[G] - L_1 >$ is not regular , a contradiction .

Conversly, suppose G has $\{L_1\} = C$ cut vertices and exactly one nonend block K_n incident with complete blocks K_{n-c+1}. Then $\{L_1\}$ is a dominating set of G. Further $< V(G) - L_1 >$ is regular with more than one component. Clearly D forms a $\gamma_{crs} - set$. Hence $|D| = |L_1|$ gives $\gamma_{crs} = C$.

Theorem13: For any graph G with C cut vertices every nonend vertex of G is adjacent with at least one end vetex then $\gamma_{crs} = C$.

Proof: For necessary condition, let $V_1 = \{v_1, v_2, v_3, \dots \dots, v_l\} \subseteq V(G)$ be set of all end vertices in G. Let $V_2 \subseteq \{V(G) - V_1\}$ forms a $\gamma - set$ of G. And let $A = \{v_1, v_2, \dots \dots \dots \dots, v_m\} \subseteq V_2$ be the set of cut vertices of G. Suppose $V_3 = \{v_1, v_2, v_3, \dots \dots, v_n\} \subseteq V_2$ be the set of nonend vertices. Then there exists at least one vertex v_i which is not adjacent to an end vertex . Since $v_j \in N(v_i)$ and $v_j \notin V_2$ and $v_i \in V_2$ then $< V(G) - V_2 >$ is disconnected and we consider the following cases.

Case1: Suppose G is a tree. Then $A = \{v_1, v_2, \dots \dots, v_n\}$ be the set of all nonend vertices which are cutvertices. Suppose there exists $V'_1 \subseteq A$ which are adjacent to end vertices of T. Now assume there exists at least one vertex $v_k \in N(V'_1)$ and $v_k \notin V_2$, since v_k is a cutvertex and $< V(T) - V_2 >$ is disconnected and regular, then $|V_2| > |V_1|$ which gives, $\gamma_{crs} \neq C$.

Case 2: Suppose G is not a tree . Then there exists at least one block which is cycle. Let v be a vertex which is not incident with an end vertex and $v \in D$ then $< V - V_2 >$ is not regular hence D is not a $\gamma_{crs} - set$ of G. Then there exists at least one vertex $u \in \{V(G) - V_2\}$ such that $< V(G) - \{V_2 \cup u\} >$ is regular and $\gamma_{crs} - set$ of G. Hence $|V_2 \cup \{u\}| > |C|$.

For sufficient conditions, let every nonend vertex of G is adjacent with at least one end vertex. Then $V_2 = \{V(G) - V_1\}$ is a dominating set of G. Also $< V(G) - V_2 >$ is disconnected and $\deg(v_i) = 0 \ \forall \ v_i \in \{V(G) - V_1\}$. Thus V_2 is $\gamma_{crs} - set$ of G . Since every vertex of V_2 is a cut vertex , then $|V_2| = |C|$. Clearly $\gamma_{crs} = C$.

REFERENCES

[1] E. J. Cockayne , R.M Dawes, S.T Hedetniemi ,Total domination in graphs, Networks 10 (1980) 211-219.

[2] F. Harary, Graph Theory Addison, Wesly Reading Mass, 1969.

[3] T. W. Haynes, S.T. Hedetniemi , and P.J.slater , Fundamentals of domination in graphs, Marcel Dekker , Inc, Newyork(1998).

[4] S.T. Hedetniemi and R.Laskeer, Connected domination in graphs , Graph Theory and Combinatorics, Cambridge (Academic press London, 1984) 209-217:Mr.86e:05055.

[5] V.R Kulli , Theory of domination in graphs ,Vishwa International publications (2010).

[6] M .H. Muddebihal, Priyanka .H .Mandarvadkar , Global domination in line graphs (2019) 648-652.

[7] E. Sampathkumar, The global domination number of graph, J. Math phy scie.23(1989) 377-385.