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AN INTEGRAL INVOLVING G-FUNCTION OF TWO VARIABLES

By

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ABSTRACT

Kandu [8], Agarwal [2], Gupta [7], Chitra [4], Agarwal [1], Gaunt [6], Ayant [3] and several other authors have evaluated some finite, infinite and double integrals involving the generalized hypergeometric functions.

Looking importance and usefulness of integral in various fields, in this paper we have establish a new integral involving G-Function of two variables, which will be helpful in the study of boundary value problems, expansion formula, Fourier series, statistical distribution, probability and integral equation.

Key Words: G-Function of two variables, Integral, Hypergeometric functions, Gamma Function.

1. INTRODUCTION:

The G-function of two variables was defined by Shrivastava and Joshi [10, p. 471] in terms of Mellin-Barnes type integrals as follows:

$$G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}\\(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}\end{matrix}\right]$$

$$= \frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\xi+\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\xi-\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\xi+\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(d_j-\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\xi)}$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3}\Gamma(f_j-\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+\eta)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-\eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, (i = 1, 2, 3; j = 2, 3).

The contour L_1 is in the ξ-plane and runs from – i∞ to + i∞, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \xi)$ (j = 1, ..., m_2) lie to the right, and the poles of $\Gamma(1 - c_j + \xi)$ (j = 1, ..., n_2), $\Gamma(1 - a_j + \xi + \eta)$ (j = 1, ..., n_1) to the left of the contour.

The countor L_2 is in the η-plane and runs from – i∞ to + i∞, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - \eta)$ (j = 1, ..., m_3) lie to the right, and the poles of $\Gamma(1 - e_j + \eta)$ (j = 1, ..., n_3), $\Gamma(1 - a_j + \xi + \eta)$ (j = 1, ..., n_1) to the left of the contour, and the double integral converges if

$$2(n_1 + m_2 + n_2) > (p_1 + q_1 + p_2 + q_2)$$

$$2(n_1 + m_3 + n_3) > (p_1 + q_1 + p_3 + q_3)$$

and $|\arg x| < \tfrac{1}{2} U\pi$, $|\arg y| < \tfrac{1}{2} V\pi$.

where $U = [n_1 + m_2 + n_2 - \tfrac{1}{2}(p_1 + q_1 + p_2 + q_2)]$

$$V = [n_1 + m_3 + n_3 - \tfrac{1}{2}(p_1 + q_1 + p_3 + q_3)]$$

These assumptions for the G-function of two variables will be adhered to throughout this research work.

The following formulae are required in the proof:

From Sharma [9, p.3-5]:

$$\int_0^\infty e^{2\mu\theta} (\sinh\theta)^{2\lambda-1} \; {}_2F_1 \left[\begin{matrix} \lambda - \mu + \frac{1}{2}, \beta; \\ \delta; \end{matrix} 2e^{-\theta}\sinh\theta\right] d\theta$$

$$= \frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{1}{4}-\frac{\lambda}{2}-\frac{\mu}{2})\Gamma(\frac{3}{4}-\frac{\lambda}{2}-\frac{\mu}{2})\Gamma(\frac{\delta}{2}-\frac{\beta}{2}-\lambda)\Gamma(\frac{1}{2}+\frac{\delta}{2}-\frac{\beta}{2}-\lambda)}{\pi 2^{\lambda+\mu+\beta+\frac{3}{2}}\Gamma(\delta-\beta)\Gamma(\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2})\Gamma(\frac{\delta}{2}-\lambda)\Gamma(\frac{1}{2}+\frac{\delta}{2}-\lambda)}, \quad (2)$$

valid for $R(\lambda) > 0$, $R(1/2 - \lambda - \mu) > 0$, $R(\delta - \beta - \lambda + \mu - 1/2) > 0$.

From Erdelyi [5, p.4, (46)]:

The multiplication formula for the Gamma-Function

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2m}} m^{mz-\frac{1}{2}} \prod_{i=1}^{m-1} \Gamma\left(z + \frac{i}{m}\right), \quad (3)$$

where m is a positive integer.

2. INTEGRAL:

Following Kandu [8], Agarwal [2], Gupta [7], Chitra [4], Agarwal [1], Gaunt [6], Ayant [3] and other authors in this section, we evaluate an integral involving G-function of two variables.

$$\int_0^\infty e^{2\mu\theta} (\sinh\theta)^{2\lambda-1} \; {}_2F_1 \left[\begin{matrix} \lambda - \mu + \frac{1}{2}, \beta; \\ \delta; \end{matrix} 2e^{-\theta}\sinh\theta\right]$$

$$\times\, G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\begin{bmatrix} z_1 e^{2m\theta}(\sinh\theta)^{2m} \\ z_2 \end{bmatrix} d\theta$$

$$= \frac{\pi^{1-2m} m^{\lambda-\mu-\frac{\beta}{2}-\frac{1}{2}}\Gamma(\delta)}{2^{\lambda+\mu+\beta-\frac{1}{2}}\Gamma(\delta-\beta)\Gamma(\frac{1}{2}+\lambda-\mu)} G^{0,\ n_1\ :m_2+4m,n_2+2m;m_3,n_3}_{p_1,q_1\ :p_2+4m,q_2+4m\ ;p_3,q_3}\left[\begin{matrix} z_1/2^{2m} \\ z_2 \end{matrix}\right|$$

$$\left.\begin{matrix} (a_j;1,1)_{1,p_1}: \phi_1,\dots,\phi_{2m}, c_1,\dots,c_{p_2}, \phi_{2m+1},\dots,\phi_{4m}: (e_j,1)_{1,p_3} \\ (b_j;1,1)_{1,q_1}: \psi_1,\dots,\psi_{4m}, d_1,\dots,d_{q_2}: (f_j,1)_{1,q_3} \end{matrix}\right], \qquad (4)$$

where m is a positive integer, and

$$\phi_{k+1} = \frac{1-\lambda+k}{m}, \phi_{k+m+1} = \frac{1-2\lambda+2k}{2m},$$
$$\phi_{k+2m+1} = \frac{1+\delta-2\lambda+2k}{2m}, \phi_{k+3m+1} = \frac{\delta-2\lambda+2k}{2m},$$
$$\psi_{k+1} = \frac{1-2\lambda-2\mu+4k}{4m}, \psi_{k+m+1} = \frac{3-2\lambda-2\mu+4k}{4m},$$
$$\psi_{k+2m+1} = \frac{\delta-\beta-2\lambda+2k}{2m}, \psi_{k+3m+1} = \frac{1-\beta+\delta-2\lambda+2k}{2m},$$
$$k = 0,1,2,\dots,(m-1).$$

The result is valid under the following conditions:

(i) $p_2 + q_2 < 2(m_2 + n_2), |\arg z_1| < \left(m_2 + n_2 - \frac{1}{2}p_2 - \frac{1}{2}q_2\right)\pi$,
$R(\delta - \beta - \lambda + \mu - 1/2) > 0$, $R(\lambda + md_j) > 0$, for j = 1, 2, …, m_2
$R(\lambda + \mu - 2m - 2mc_h - 1/2) < 0$, for h = 1, 2, …, n_2.

(ii) $p_2 \le q_2, p_2 + q_2 \le 2(m_2 + n_2), |\arg z_1| < \left(m_2 + n_2 - \frac{1}{2}p_2 - \frac{1}{2}q_2\right)\pi$,
$R(\delta - \beta - \lambda + \mu - 1/2) > 0$, $R(\lambda + md_j) > 0$, for j = 1, 2, …, m_2
$R(\lambda + \mu - 2mc_h) < \frac{4m+1}{2}$, for h = 1, 2, …, n_2.
$2m\left(\sum_{j=1}^{p_2} c_j - \sum_{j=1}^{q_2} d_j - \frac{1}{2}\right) - (q_2 - p_2)\left(\lambda + \mu - m - \frac{1}{2}\right) > 0$

(iii) $p_2 \ge q_2, p_2 + q_2 \le 2(m_2 + n_2), |\arg z_1| < \left(m_2 + n_2 - \frac{1}{2}p_2 - \frac{1}{2}q_2\right)\pi$,
$R(\delta - \beta - \lambda + \mu - 1/2) > 0$, $R(\lambda + md_j) > 0$, for j = 1, 2, …, m_2
$R(\lambda + \mu - 2mc_h) < \frac{4m+1}{2}$, for h = 1, 2, …, n_2.
$2m\left(\sum_{j=1}^{p_2} c_j - \sum_{j=1}^{q_2} d_j + \frac{1}{2}\right) + (p_2 - q_2)(\lambda - m) > 0.$

Proof of (4):

To prove (4), we substitute the contour integral (1) for the G-function of two variables in the integrand of (4), change the order of integration (which we suppose to be permissible) and evaluate the inner integral with the help of (2).

The value of the integral then becomes

$$\frac{2^{-\lambda-\mu-\beta-\frac{3}{2}}\Gamma(\delta)}{\Gamma(\delta-\beta)\Gamma\left(\frac{1}{2}+\lambda-\mu\right)\pi} \frac{-1}{4\pi^2} \int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)$$

$$\times \frac{\Gamma(\lambda+m\xi)\Gamma\left(\frac{1}{2}+\lambda+m\xi\right)\Gamma\left(\frac{1}{4}-\frac{\lambda}{2}-\frac{\mu}{2}-m\xi\right)\Gamma\left(\frac{3}{4}-\frac{\lambda}{2}-\frac{\mu}{2}-m\xi\right)}{\Gamma\left(\frac{\delta}{2}-\lambda-m\xi\right)}$$
$$\times \frac{\Gamma\left(\frac{\delta}{2}-\frac{\beta}{2}-\lambda-m\xi\right)\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\frac{\beta}{2}-\lambda-m\xi\right)}{\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\lambda-m\xi\right)}\left[2^{-2m}z_1\right]^{\xi}{z_2}^{\eta}d\xi d\eta$$
$$= \frac{m^{\lambda-\mu-\frac{\beta}{2}-\frac{1}{2}}\pi^{1-2m}\Gamma(\delta)}{2^{\lambda+\mu+\beta-\frac{1}{2}}\Gamma(\delta-\beta)\Gamma\left(\frac{1}{2}+\lambda-\mu\right)}\frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)$$
$$\times \frac{\prod_{k=0}^{m-1}\Gamma\left(\frac{k+\lambda}{m}+\xi\right)\prod_{k=0}^{m-1}\Gamma\left(\frac{1+2\lambda+2k}{2m}+\xi\right)\prod_{k=0}^{m-1}\Gamma\left(\frac{1-2\lambda-2\mu+4k}{4m}-\xi\right)}{\prod_{k=0}^{m-1}\Gamma\left(\frac{1+\delta-2\lambda+2k}{2m}-\xi\right)}$$
$$\times \frac{\prod_{k=0}^{m-1}\Gamma\left(\frac{3-2\lambda-2\mu+4k}{4m}-\xi\right)\prod_{k=0}^{m-1}\Gamma\left(\frac{\delta-\beta-2\lambda+2k}{2m}-\xi\right)\prod_{k=0}^{m-1}\Gamma\left(\frac{1+\delta-2\lambda-\beta+2k}{2m}-\xi\right)}{\prod_{k=0}^{m-1}\Gamma\left(\frac{\delta-2\lambda+2k}{2m}-\xi\right)}$$
$$\times \left[2^{-2m}z_1\right]^{\xi}{z_2}^{\eta}d\xi d\eta, \tag{5}$$

by virtue of (3).

The contour L_1 and L_2 runs – i∞ to + i∞, so that the poles of $\Gamma(d_j-\xi)$ for j = 1, 2, …, m_2, $\Gamma\left(\frac{1-2\lambda-2\mu+4k}{4m}-\xi\right)$, $\Gamma\left(\frac{3-2\lambda-2\mu+4k}{4m}-\xi\right)$, $\Gamma\left(\frac{\delta-\beta-2\lambda+2k}{2m}-\xi\right)$ and $\Gamma\left(\frac{1+\delta-2\lambda-\beta+2k}{2m}-\xi\right)$ for k = 0, 1, 2, …, (m − 1) are to the right, and all the poles of $\Gamma(1-c_j+\xi)$ for j = 1, 2, …, n_2, $\Gamma\left(\frac{k+\lambda}{m}+\xi\right)$ and $\Gamma\left(\frac{1+2\lambda+2k}{2m}+\xi\right)$, for k = 0, 1, 2, …, (m − 1) are to the right of L_1.

Interpreting (5), with the help of (1), we get (4) under the conditions stated there in.

3. SPECIAL CASES:

On specializing the parameters in (5) we get following integral in terms of G-function of one variable:

$$\int_0^{\infty} e^{2\mu\theta}(\sinh\theta)^{2\lambda-1}\ {}_2F_1\left[\begin{matrix}\lambda-\mu+\frac{1}{2},\beta;\\ \delta;\end{matrix} 2e^{-\theta}\sinh\theta\right]$$
$$\times G_{r,s}^{p,q}\left[e^{2m\theta}(\sinh\theta)^{2m}z\Big|_{(b_j,1)_{1,s}}^{(a_j,1)_{1,r}}\right]d\theta$$
$$= \frac{\pi^{1-2m}m^{\lambda-\mu-\frac{\beta}{2}-\frac{1}{2}}\Gamma(\delta)}{2^{\lambda+\mu+\beta-\frac{1}{2}}\Gamma(\delta-\beta)\Gamma(\frac{1}{2}+\lambda-\mu)}G_{r+4m,s+4m}^{p+4m,q+2m}\left[z^{2m}\Big|\begin{matrix}\phi_1,\dots,\phi_{2m},a_1,\dots,a_r,\phi_{2m+1},\dots,\phi_{4m}\\ \psi_1,\dots,\psi_{4m},b_1,\dots,b_s\end{matrix}\right], \tag{6}$$

where m is a positive integer, and

$$\phi_{k+1}=\frac{1-\lambda+k}{m},\phi_{k+m+1}=\frac{1-2\lambda+2k}{2m},$$

$\phi_{k+2m+1} = \frac{1+\delta-2\lambda+2k}{2m}, \phi_{k+3m+1} = \frac{\delta-2\lambda+2k}{2m},$
$\psi_{k+1} = \frac{1-2\lambda-2\mu+4k}{4m}, \psi_{k+m+1} = \frac{3-2\lambda-2\mu+4k}{4m},$
$\psi_{k+2m+1} = \frac{\delta-\beta-2\lambda+2k}{2m}, \psi_{k+3m+1} = \frac{1-\beta+\delta-2\lambda+2k}{2m},$
$k = 0,1,2, \dots, (m-1).$

The result is valid under the following conditions:

(i) $r + s < 2(p+q), |\arg z| < \left(p + q - \frac{1}{2}r - \frac{1}{2}s\right)\pi,$
$R(\delta - \beta - \lambda + \mu - 1/2) > 0$, $R(\lambda + mb_j) > 0$, for j = 1, 2, …, p
$R(\lambda + \mu - 2m - 2ma_h - 1/2) < 0$, for h = 1, 2, …, q.

(ii) $r \leq s, r + s \leq 2(p+q), |\arg z| < \left(p + q - \frac{1}{2}r - \frac{1}{2}s\right)\pi,$
$R(\delta - \beta - \lambda + \mu - 1/2) > 0$, $R(\lambda + mb_j) > 0$, for j = 1, 2, …, p
$R(\lambda + \mu - 2ma_h) < \frac{4m+1}{2}$, for h = 1, 2, …, q.
$2m\left(\sum_{j=1}^{r} a_j - \sum_{j=1}^{s} b_j - \frac{1}{2}\right) - (s - r)\left(\lambda + \mu - m - \frac{1}{2}\right) > 0$

(iii) $r \geq s, r + s \leq 2(p+q), |\arg z| < \left(p + q - \frac{1}{2}r - \frac{1}{2}s\right)\pi,$
$R(\delta - \beta - \lambda + \mu - 1/2) > 0$, $R(\lambda + mb_j) > 0$, for j = 1, 2, …, p
$R(\lambda + \mu - 2ma_h) < \frac{4m+1}{2}$, for h = 1, 2, …, q.
$2m\left(\sum_{j=1}^{r} a_j - \sum_{j=1}^{s} b_j + \frac{1}{2}\right) + (r - s)(\lambda - m) > 0.$

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